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BULGING DEFORMATIONS OF CONVEX $\mathbb{RP}^2$-MANIFOLDS

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Abstract. We define deformations of convex $\mathbb{RP}^2$-surfaces.

A *convex* $\mathbb{RP}^2$*-manifold* is a representation of a surface S as a quotient Ω/Γ, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathsf{SL}(3,\mathbb{R})$ is a discrete group of collineations acting properly on Ω. We shall describe a construction of deformations of such structures based on Thurston's earthquake deformations for hyperbolic surfaces and *quakebend deformations* for $\mathbb{CP}^1$-manifolds.

In general if Ω/Γ is a convex $\mathbb{RP}^2$-manifold which is a *closed* surface S with $\chi(S)$, then either $\partial\Omega$ is a conic, or $\partial\Omega$ is a C^1 convex curve (Benzécri [1]) which is not C^2 (Kuiper [5]). In fact its derivative is Hölder continuous with Hölder exponent strictly between 1 and 2. Figure 1 depicts such a domain tiled by the $(3,3,4)$-triangle tesselation.

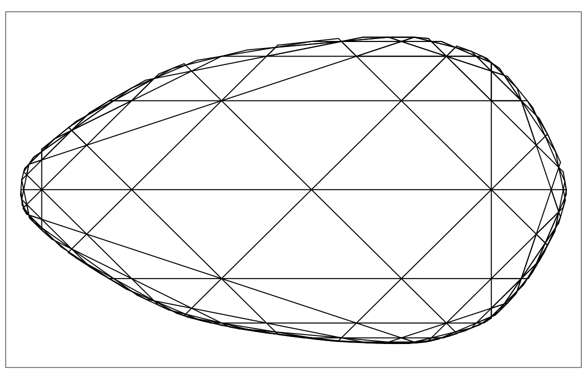

Figure 1. A convex domain tiled by triangles

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This drawing actually arises from Lie algebras (see Kac-Vinberg [4]). Namely the Cartan matrix

$$C = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

determines a group of reflections as follows. For $i = 1, 2, 3$ let E_{ii} denote the elementary matrix having entry 1 in the i-th diagonal slot. Then, for $i = 1, 2, 3$, the reflections

$$\rho_i = I - E_{ii} C$$

generate a discrete subgroup of $\mathsf{SL}(3, \mathbb{Z})$ which acts properly on the convex domain depicted in Figure 1 (and appears on the cover of the November 2002 Notices of the American Mathematical Society).. This group is the Weyl group of a hyperbolic Kac-Moody Lie algebra.

We describe here a general construction of such convex domains as limits of *piecewise conic* curves.

If Ω/Γ is a convex $\mathbb{RP}^2$-manifold homeomorphic to a closed esurface S with $\chi(S) < 0$, then every element $\gamma \in \Gamma$ is *positive hyperbolic,* that is, conjugate in $\mathsf{SL}(3, \mathbb{R})$ to a diagonal matrix of the form

$$\delta = \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-s-t} \end{bmatrix}.$$

where $s > t > -s - t$. Its centralizer is the *maximal* $\mathbb{R}$*-split torus* $\mathbb{A}$ consisting of all diagonal matrices in $\mathsf{SL}(3, \mathbb{R})$. It is isomorphic to a Cartesian product $\mathbb{R}^* \times \mathbb{R}^*$ and has four connected components. Its identity component $\mathbb{A}^+$ consists of diagonal matrices with positive entries.

The *roots* are linear functionals on its Lie algebra $\mathfrak{a}$, the *Cartan subalgebra.* Namely, $\mathfrak{a}$ consists of diagonal matrices

$$a = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}. \tag{0.0.1}$$

where $a_1 + a_2 + a_3 = 0$. The roots are the six linear functionals on $\mathfrak{a}$ defined by

$$a \xmapsto{\alpha_{ij}} a_i - a_j$$

where $1 \leq i \neq j \leq 3$. Evidently $\alpha_{ji} = -\alpha_{ij}$.

Writing $a(s, t)$ for the diagonal matrix (0.0.1) with

$$a_1 = s, \qquad a_1 = 2, \qquad a_3 = -s - t,$$

the roots are the linear functionals defined by

$$\begin{aligned}
a(s,t) &\xmapsto{\alpha_{12}} s - t \\
a(s,t) &\xmapsto{\alpha_{21}} t - s \\
a(s,t) &\xmapsto{\alpha_{23}} t - (-s-t) = s + 2t \\
a(s,t) &\xmapsto{\alpha_{32}} (-s-t) - t = -s - 2t \\
a(s,t) &\xmapsto{\alpha_{31}} (-s-t) - s = -2s - t \\
a(s,t) &\xmapsto{\alpha_{13}} s - (-s-t) = 2s + t
\end{aligned}$$

which we write as

$$\begin{aligned}
\alpha_{12} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \\
\alpha_{21} &= \begin{bmatrix} -1 & 1 \end{bmatrix} \\
\alpha_{23} &= \begin{bmatrix} 1 & 2 \end{bmatrix} \\
\alpha_{32} &= \begin{bmatrix} -1 & -2 \end{bmatrix} \\
\alpha_{31} &= \begin{bmatrix} -2 & -1 \end{bmatrix} \\
\alpha_{13} &= \begin{bmatrix} 2 & 1 \end{bmatrix}
\end{aligned}$$

The *Weyl group* is generated by reflections in the roots and in this case is just the symmetric group, consisting of permutations of the three variables a_1, a_2, a_3 in a (as in (0.0.1)). A fundamental domain is the *Weyl chamber* consisting of all a satisfying $\alpha_{12} > 0$ and $\alpha_{23} > 0$. This corresponds to the ordering of the roots where $\alpha_{12} > \alpha_{23}$ are the *positive simple roots.* In other words, the roots are totally ordered by: the rule

$$\alpha_{13} > \alpha_{12} > \alpha_{23} > 0 > \alpha_{32} > \alpha_{21} > \alpha_{21} > \alpha_{31}.$$

In terms of the parametrization of $\mathfrak{a}$ by $a(s,t)$, the Weyl chamber equals

$$\{a(s,t) \mid s \geq t \geq -\frac{1}{2}s\}.$$

The *trace form* on $\mathfrak{sl}(3,\mathbb{R})$ defines the inner product $\langle , \rangle$ with associated quadratic form

$$\mathsf{tr}\big(a(s,t)^2\big) = 2(s^2 + st + t^2) = 2|s + \omega t|^2$$

where $\omega = \frac{1}{2} + \frac{\sqrt{-3}}{2} = e^{\pi i/3}$ is the primitive sixth root of 1.

The elements of $\mathfrak{sl}(3,\mathbb{R})$ which dual to the roots (via the inner product $\langle,\rangle$) are the *root vectors:*

$$\begin{aligned} h_{12} &= a(1,-1),\\ h_{21} &= a(-1,1),\\ h_{23} &= a(0,1),\\ h_{32} &= a(0,-1),\\ h_{31} &= a(-1,0),\\ h_{13} &= a(1,0) \end{aligned}$$

The Weyl chamber consists of all

$$a(s,\lambda s) = \begin{bmatrix} s & 0 & 0\\ 0 & \lambda s & 0\\ 0 & 0 & -(1+\lambda)s \end{bmatrix}$$

where $1 \geq \lambda \geq -\frac{1}{2}$. Its boundary consists of the rays generated by the *singular elements*

$$a(1,1) = h_{13} + h_{23} = h_{12} + 2h_{23}$$

and

$$a(2,-1) = h_{12} + h_{13} = 2h_{12} + h_{23}.$$

The sum of the simple positive roots is the element

$$a(1,0) = h_{13} = h_{12} + h_{23}$$

which generates the one-parameter subgroup

$$H_t := \exp\big(a(t,0)\big) = \begin{bmatrix} e^t & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-t} \end{bmatrix}.$$

The orbits of $\mathbb{A}^+$ on $\mathbb{RP}^2$ are the four open 2-simplices defined by the homogeneous coordinates, their (six) edges and their (three) vertices. The orbits of H_t are arcs of conics depicted in Figure 2.

Associated to any measured geodesic lamination λ on a hyperbolic surface S is *bulging deformation* as an $\mathbb{RP}^2$-surface. Namely, one applies a one-parameter group of collineations

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & e^t & 0\\ 0 & 0 & 1 \end{bmatrix}$$

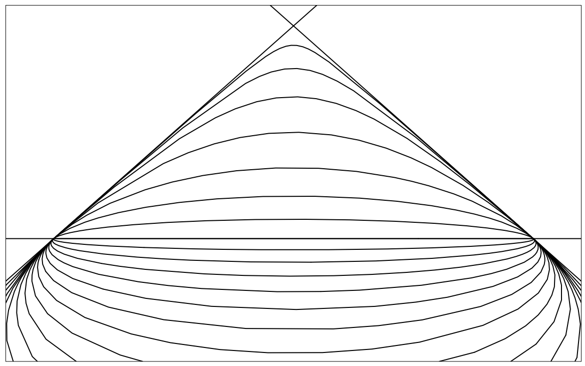

FIGURE 2. Conics tangent to a triangle

to the coordinates on either side of a leaf. This extends Thurston's *earthquake* deformations (the analog of *Fenchel-Nielsen twist deformations* along possibly infinite geodesic laminations), and the *bending deformations* in $\mathsf{PSL}(2,\mathbb{C})$.

In general, if S is a convex $\mathbb{RP}^2$-manifold, then deformations are determined by a geodesic lamination with a transverse measure taking values in the Weyl chamber of $\mathsf{SL}(3,\mathbb{R})$. When S is itself a hyperbolic surface, all the deformations in the singular directions become earthquakes and deform $\partial\tilde{S}$ trivially (just as in $\mathsf{PSL}(2,\mathbb{C})$.

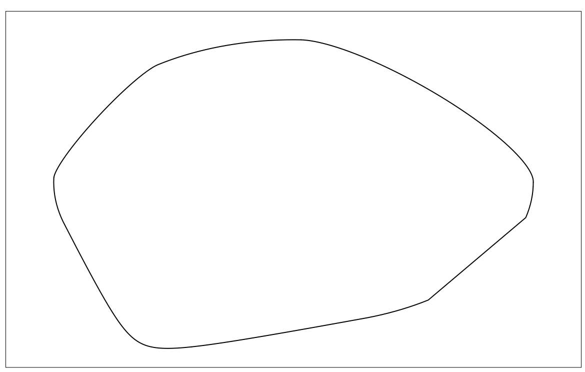

FIGURE 3. Deforming a conic

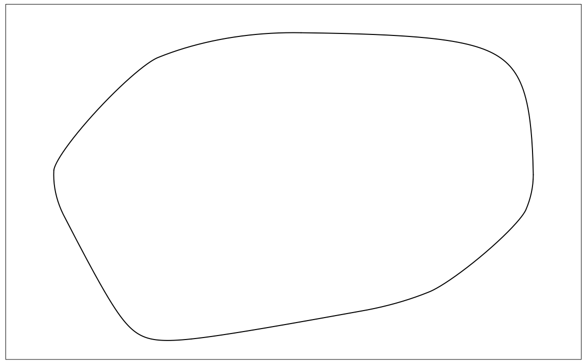

FIGURE 4. A piecewise conic

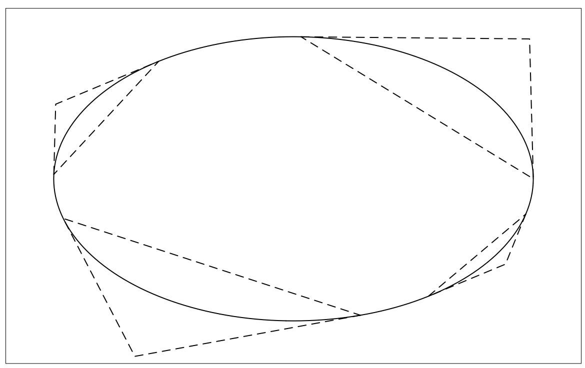

FIGURE 5. Bulging data

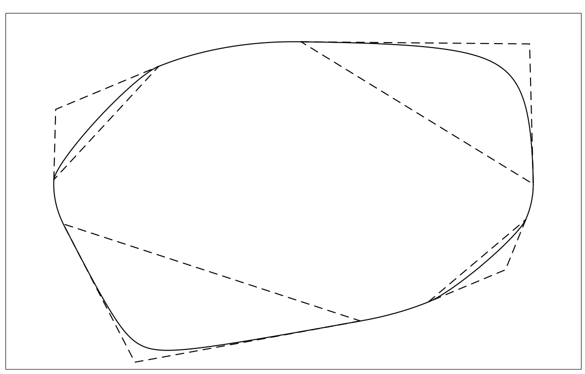

FIGURE 6. The deformed conic

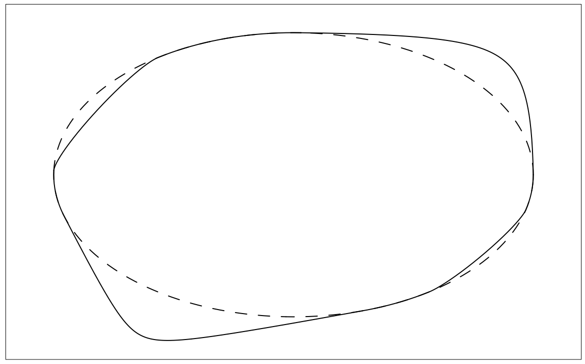

FIGURE 7. The conic with its deformation

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